Geometric Flow appearing in Conservation Law in Classical and Quantum Mechanics

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The appearance of a geometric flow in the conservation law of particle number in classical particle diffusion and in the conservation law of probability in quantum mechanics is discussed in the geometrical environment of a two-dimensional curved surface with thickness ϵ embedded in R_3. In such a system with a small thickness ϵ, the usual two-dimensional conservation law does not hold and we find an anomaly. The anomalous term is obtained by the expansion of ϵ. We find that this term has a Gaussian and mean curvature dependence and can be written as the total divergence of some geometric flow. We then have a new conservation law by adding the geometric flow to the original one. This fact holds in both classical and quantum mechanics when we confine particles to a curved surface with a small thickness.

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1. Introduction

The motion of particles on a given curved surface M^2 is an interesting problem in a wide range fields in physics [1]-[6]. The classical diffusion equation and the Schrödinger equation on such a manifold is expressed by changing the Laplacian to the Laplace-Beltrami operator in the equation; however, when the surface has a thickness ϵ, i.e., the configuration space is $M^2 \times R^1$, the situation is not simple [2], [4].

Such a 2+1 dimensional system can be found as the motion of a protein in a lipid-bilayer (cell membrane) in classical mechanics [1]. The direct application to the quantum mechanical particle is not yet done, however, the motion of an electron in the quantum hall device may be a good example to test the geometric effect by bending the surface of the system. Though it is not for the real particle, such a geometrical effect is discussed by several authors for the Josephson junction device [7]. The curvature of junction surface gives the effect to the kink motion that is the solution of sine-Gordon equation, the effective theory for the phase difference between superconducting electrodes.

In this paper we consider the conservation law in such a pseudo two dimensional geometry and show the existence of anomalous term. The case of classical mechanics is already discussed in the previous paper [2], [3], so we just give the result and we devote to discuss the quantum mechanical case.

2. Geometrical Tools

To make the problem concrete, we first introduce a two-dimensional curved manifold Σ in R^3, and we also introduce two similar copies of Σ denoted Σ' and $\tilde{\Sigma}$ and place them on both sides of Σ at a small distance of $\epsilon/2$.

Our physical space is between these two surfaces Σ' and $\tilde{\Sigma}$. We set the coordinate system as follows.

$\vec{X}$ is a Cartesian coordinate in R_3. $\vec{x}$ is a Cartesian coordinate that specifies only the points on Σ. q^i is a curved coordinate on Σ, where the small Latin indices $i, j, k, \cdots$ run from 1 to 2. q^0 is the coordinate in R_3 normal to Σ. Furthermore by using the normal unit vector $\vec{n}(q^1, q^2)$ on Σ at the point (q^1, q^2), we can identify any point between the two surfaces Σ' and $\tilde{\Sigma}$ by the following thin-layer approximation [6]:

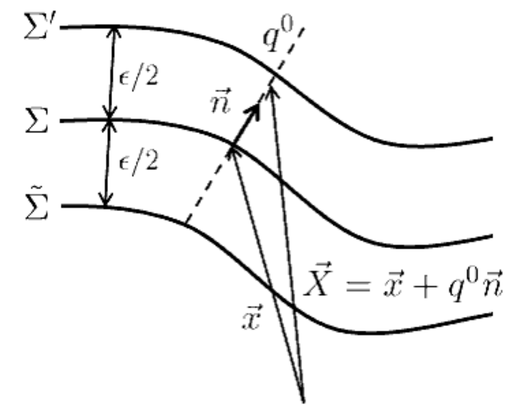

FIG. 1: curved surface with thickness ϵ

$$\vec{X}(q^0, q^1, q^2) = \vec{x}(q^1, q^2) + q^0\vec{n}(q^1, q^2), \tag{1}$$

where $-\epsilon/2 \le q^0 \le \epsilon/2$.

Then we obtain the curvilinear coordinate system between two surfaces ($\subset R_3$) by using the coordinate $q^\mu = (q^0, q^1, q^2)$ and the metric $G_{\mu\nu}$. (Hereafter, Greek indices $\mu, \nu, \cdots$ run from 0 to 2.)

$$G_{\mu\nu} = \frac{\partial \vec{X}}{\partial q^\mu} \cdot \frac{\partial \vec{X}}{\partial q^\nu}. \tag{2}$$

Each part of $G_{\mu\nu}$ is expressed as follows:

$$G_{ij} = g_{ij} + q^0\left(\frac{\partial \vec{x}}{\partial q^i}\cdot\frac{\partial \vec{n}}{\partial q^j} + \frac{\partial \vec{x}}{\partial q^j}\cdot\frac{\partial \vec{n}}{\partial q^i}\right) + (q^0)^2\frac{\partial \vec{n}}{\partial q^i}\cdot\frac{\partial \vec{n}}{\partial q^j}, \tag{3}$$

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where

$$g_{ij}=\frac{\partial\vec{x}}{\partial q^i}\cdot\frac{\partial\vec{x}}{\partial q^j} \tag{4}$$

is the metric (first fundamental tensor) on Σ. Hereafter, the indices $i,j,k\cdots$ are lowered or raised by g_{ij} and its inverse g^{ij}. We also obtain

$$G_{0i}=G_{i0}=0,\quad G_{00}=1. \tag{5}$$

We can perform the calculation by using new variables. We first define the tangential vector to Σ by

$$\vec{B}_k=\frac{\partial\vec{x}}{\partial q^k}. \tag{6}$$

Note that $\vec{n}\cdot\vec{B}_k=0$. Then we obtain two relations: the Gauss equation

$$\frac{\partial\vec{B}_i}{\partial q^j}=-\kappa_{ij}\vec{n}+\Gamma^k_{ij}\vec{B}_k \tag{7}$$

and the Weingarten equation

$$\frac{\partial\vec{n}}{\partial q^j}=\kappa^m_j\vec{B}_m, \tag{8}$$

where

$$\Gamma^k_{ij}\equiv\frac{1}{2}g^{km}(\partial_i g_{mj}+\partial_j g_{im}-\partial_m g_{ij}).$$

κ_{ij} is a symmetric tensor called the Euler-Schauten tensor, or the second fundamental tensor defined by the above two equations. Furthermore, the mean curvature is given by

$$\kappa=g^{ij}\kappa_{ij}, \tag{9}$$

and the Ricci scalar R (Gaussian curvature) is defined by

$$R/2\equiv\det(g^{ik}\kappa_{kj})=\det(\kappa^i_j)=\frac{1}{2}(\kappa^2-\kappa_{ij}\kappa^{ij}). \tag{10}$$

Then we have the following formula for the metric of curvilinear coordinates in a neighborhood of Σ:

$$G_{ij}=g_{ij}+2q^0\kappa_{ij}+(q^0)^2\kappa_{im}\kappa^m_j. \tag{11}$$

Now we have a total metric tensor such as

$$G_{\mu\nu}=\begin{pmatrix}1 & 0\\ 0 & G_{ij}\end{pmatrix}. \tag{12}$$

From the definition of the metric (11) and the Ricci scalar (10), we can construct the following geometrical quantities:

$$G\equiv\det G_{ij}=g+2g\kappa q^0+g(\kappa^2+R)(q^0)^2+\mathcal{O}((q^0)^3). \tag{13}$$

The inverse metric of G_{ij} is given as

$$G^{ij}=g^{ij}-2\kappa^{ij}q^0+\frac{3}{2}(2\kappa\kappa^{ij}-Rg^{ij})(q^0)^2+\mathcal{O}((q^0)^3). \tag{14}$$

Furthermore,

$$(\kappa^{-1})^{ij}=\frac{2}{R}(\kappa g^{ij}-\kappa^{ij}), \tag{15}$$

and from this relationship we obtain

$$\frac{1}{2}Rg^{ij}=\kappa\kappa^{ij}-\kappa^i_m\kappa^{mj}. \tag{16}$$

By using the above relations, we can construct the diffusion equation and Schrödinger equation in our environment.

3. Effective Schrödinger equation and Geometric Flow

For the classical diffusion field, the problem has already been solved and discussed in [2]. We just note the important result.

$$\begin{aligned}\frac{\partial\phi^{(2)}}{\partial t} &= D\Delta^{(2)}\phi^{(2)}+\tilde{D}g^{-1/2}\frac{\partial}{\partial q^i}\ g^{1/2}\\ &\times\ \{(3\kappa^{im}\kappa^j_m-2\kappa\kappa^{ij})\frac{\partial}{\partial q^j}-\frac{1}{2}g^{ij}\frac{\partial R}{\partial q^j}\}\phi^{(2)}\\ &= -\nabla^{(2)}_i(J^i+J^i_G),\end{aligned} \tag{17}$$

where D, $\tilde{D}$, $\phi^{(2)}$ and $\nabla^{(2)}_i$ are the diffusion constant, $\tilde{D}\equiv\frac{\epsilon^2}{12}D$, the two-dimensional effective diffusion field and the two-dimensional covariant derivative respectively. The normal diffusion flow J is

$$J^i=-Dg^{ij}\frac{\partial\phi^{(2)}}{\partial q^j}, \tag{18}$$

and the geometric diffusion flow J_G is

$$J^i_G=-\tilde{D}\ [\ (3\kappa^{im}\kappa^j_m-2\kappa\kappa^{ij})\frac{\partial\phi^{(2)}}{\partial q^j}-\frac{1}{2}g^{ij}\frac{\partial R}{\partial q^j}\phi^{(2)}]. \tag{19}$$

The two conditions that the surface is curved, and $\epsilon\neq 0$ are essential for the existence of geometric flow.

In quantum mechanics, the basic equation is the Schrödinger equation which is written using curvilinear coordinates in a three-dimensional space between the two surfaces Σ' and $\tilde{\Sigma}$.

$$i\hbar\frac{\partial}{\partial t}\psi=[-\frac{\hbar^2}{2m}\Delta^{(3)}+V(q)]\psi, \tag{20}$$

where the form of the Laplace-Beltrami operator is

$$\Delta^{(3)} \equiv G^{-1/2}\partial_\mu G^{1/2} G^{\mu\nu}\partial_\nu,$$

and we suppose that V depends on neither t nor q^0.

Starting from this wave function ψ, we construct the effective two-dimensional theory [4],[5],[6]. From the normalization condition we obtain

$$1 = \int |\psi|^2 \sqrt{G} d^3q = \int [\int_{-\epsilon/2}^{+\epsilon/2} |\psi|^2 \sqrt{\frac{G}{g}} \ \ dq^0] \sqrt{g} d^2q. \tag{21}$$

Our effective two-dimensional wave function φ should satisfy

$$\begin{aligned} |\varphi(q^1,q^2)|^2 &= \int_{-\epsilon/2}^{+\epsilon/2} |\psi|^2 \sqrt{\frac{G}{g}} \ \ dq^0, & (22)\\ 1 &= \int |\varphi|^2 \sqrt{g}\ d^2q. & (23)\end{aligned}$$

Then how can we obtain the dynamical equation for φ ? To solve this problem, we first define a new variable $\tilde{\psi}$ as

$$\tilde{\psi} \equiv (G/g)^{1/4}\psi \tag{24}$$

with

$$|\varphi(q^1,q^2)|^2 = \int_{-\epsilon/2}^{+\epsilon/2} |\tilde{\psi}|^2 \ \ dq^0, \tag{25}$$

Furthermore we suppose that it is possible to separate the variables. (Later we prove the possibility of separation of variable up to second order of ϵ expansion.)

$$\begin{aligned} \tilde{\psi} &= \varphi(q^1,q^2,t)\ \chi(q^0,t), & (26)\\ 1 &= \int_{-\epsilon/2}^{+\epsilon/2} |\chi|^2\ dq^0. & (27)\end{aligned}$$

Then we can construct the equation for φ from the Schrödinger equation of $\tilde{\psi}$. To start this program, we first construct the Schrödinger equation for $\tilde{\psi}$. This has the same form as (20) except that the Laplace-Beltrami operator is changed to the following operator.

$$\tilde{\Delta}^{(3)} \equiv (G/g)^{1/4}\Delta^{(3)}(G/g)^{-1/4}. \tag{28}$$

Using some tools prepared in section 2, this operator can be expanded as

$$\begin{aligned} \tilde{\Delta}^{(3)} &= \Delta^{(2)} + \frac{\partial^2}{\partial(q^0)^2} + V_0 + q^0V_1 + (q^0)^2V_2\\ &\quad + q^0\hat{A}_1 + (q^0)^2\hat{A}_2 + \mathcal{O}((q^0)^3), & (29)\end{aligned}$$

where $V_0, V_1, V_2, \hat{A}_1$, and $\hat{A}_2$ are given by

$$\begin{aligned} V_0 &= \frac{1}{4}(\kappa^2 - 2R), & (30)\\ V_1 &= \kappa(R - \frac{\kappa^2}{2}) - \frac{1}{2}(\Delta^{(2)}\kappa), & (31)\\ V_2 &= \frac{3}{4}\kappa^4 - \frac{7}{4}\kappa^2 R + \frac{1}{2}R^2 + \frac{1}{2}\kappa(\Delta^{(2)}\kappa)\\ &\quad + \frac{1}{4}g^{ij}(\partial_i\kappa)(\partial_j\kappa) + \{\nabla_i(\kappa^{ij}\partial_j\kappa)\}\\ &\quad - \frac{1}{4}(\Delta^{(2)}R), & (32)\\ \hat{A}_1 &= -2\nabla_i\kappa^{ij}\partial_j, & (33)\\ \hat{A}_2 &= 3\nabla_i\kappa^{ik}\kappa_k^j\partial_j. & (34)\end{aligned}$$

Then our equation for $\tilde{\psi}$ is given as follows:

$$\begin{aligned} i\hbar\frac{\partial\tilde{\psi}}{\partial t} &= -\frac{\hbar^2}{2m}[\Delta^{(2)} + \frac{\partial^2}{\partial(q^0)^2} + V_0 + q^0V_1 + (q^0)^2V_2\\ &\quad + q^0\hat{A}_1 + (q^0)^2\hat{A}_2]\tilde{\psi} + V\tilde{\psi}, & (35)\end{aligned}$$

where we have omitted $\mathcal{O}((q^0)^3)$ terms for small q^0.

We treat this system by the perturbation method. The Hamiltonian can be written as

$$\begin{aligned} \hat{H} &= \hat{H}_0 + \hat{H}_I, & (36)\\ \hat{H}_0 &= -\frac{\hbar^2}{2m}[\Delta^{(2)} + \frac{\partial^2}{\partial(q^0)^2} + V_0] + V, & (37)\\ \hat{H}_I &= -\frac{\hbar^2}{2m}[q^0(V_1 + \hat{A}_1) + (q^0)^2(V_2 + \hat{A}_2)]. & (38)\end{aligned}$$

For simplicity we write

$$\begin{aligned} \xi &\equiv q^0/\epsilon, & (39)\\ y &\equiv (q^1,q^2), & (40)\\ \hat{h}(y,\partial_y) &\equiv -\frac{\hbar^2}{2m}[\Delta^{(2)} + V_0] + V, & (41)\\ \hat{f}(y,\partial_y) &\equiv -\frac{\hbar^2}{2m}(V_1 + \hat{A}_1), & (42)\\ \hat{g}(y,\partial_y) &\equiv -\frac{\hbar^2}{2m}(V_2 + \hat{A}_2). & (43)\\ & & (44)\end{aligned}$$

Furthermore we utilise $\hbar^2/m = 1$ unit in a while. Then Hamiltonian can be written as

$$\hat{H} = -\frac{1}{2\epsilon^2}\partial_\xi^2 + \hat{h}(y,\partial_y) + \epsilon\xi\hat{f}(y,\partial_y) + \epsilon^2\xi^2\hat{g}(y,\partial_y). \tag{45}$$

The boundary condition in ξ direction is

$$\tilde{\psi}(\xi = \pm 1/2) = 0. \tag{46}$$

We look for a wave function in the form

$$\tilde{\psi} = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \cdots, \tag{47}$$

and energy eigen value

$$E = \epsilon^{-2}E_{(-2)} + \epsilon^{-1}E_{(-1)} + E_{(0)} + \epsilon E_{(1)} + \epsilon^2 E_{(2)} + \cdots. \tag{48}$$

Putting all expansions into the time independent Schrödinger equation, and we obtain the following equations in each order of ϵ.

$$-\frac{1}{2}\partial_\xi^2\psi_0 = E_{(-2)}\psi_0, \tag{49}$$

$$-\frac{1}{2}\partial_\xi^2\psi_1 = E_{(-2)}\psi_1 + E_{(-1)}\psi_0, \tag{50}$$

$$-\frac{1}{2}\partial_\xi^2\psi_2 + \hat{h}\psi_0 = E_{(0)}\psi_0 + E_{(-1)}\psi_1 + E_{(-2)}\psi_2, \tag{51}$$

$$-\frac{1}{2}\partial_\xi^2\psi_3 + \hat{h}\psi_1 + \xi\hat{f}\psi_0 = E_{(1)}\psi_0 + E_{(0)}\psi_1 + E_{(-1)}\psi_2 + E_{(-2)}\psi_3, \tag{52}$$

$$-\frac{1}{2}\partial_\xi^2\psi_4 + \hat{h}\psi_2 + \xi\hat{f}\psi_1 + \xi^2\hat{g}\psi_0 = E_{(2)}\psi_0 + E_{(1)}\psi_1 + E_{(0)}\psi_2 + E_{(-1)}\psi_3 + E_{(-2)}\psi_4. \tag{53}$$

The first equation (49) is easily solved by using boundary condition (46) and we obtain

$$\chi_N = \begin{cases} \sqrt{2}\cos(N\pi\xi) & N = \text{odd}, \\ \sqrt{2}\sin(N\pi\xi) & N = \text{even}, \end{cases} \tag{54}$$

$$E_N = \frac{\pi^2}{2}N^2, \tag{55}$$

with normalization condition

$$\int_{-1/2}^{1/2} |\chi_N|^2 \, d\xi = 1. \tag{56}$$

So we have

$$\psi_0 = a_0(y)\chi_N(\xi), \quad E_{(-2)} = E_N. \tag{57}$$

Hereafter we utilise the orthonormal condition for χ,

$$\int_{-1/2}^{1/2} \chi_M(\xi)\chi_N(\xi)d\xi = \delta_{MN}. \tag{58}$$

Note that the boundary condition (46) is an idealized condition because it is derived by the infinitely deep square well potential. Therefore (54) - (58) are the approximated relation by using a deep enough but finite confining potential.

In the same way, we suppose the existence of the eigen equation for y direction i.e.

$$\hat{h}\ \varphi_n(y) = \lambda_n\ \varphi_n(y), \tag{59}$$

and we suppose the the orthonormal condition for φ,

$$\int \varphi_m^*(y)\varphi_n(y)\sqrt{g}\ d^2y = \delta_{mn}. \tag{60}$$

From equation (50), we have

$$(-\frac{1}{2}\partial_\xi^2 - E_N)\psi_1 = E_{(-1)}a_0\chi_N.$$

We expand ψ_1 as

$$\psi_1 = \sum_M a_M^{(1)}(y)\chi_M(\xi).$$

and by putting this into the previous equation and by using (58), we obtain

$$a_K^{(1)}(E_K - E_N) = a_0E_{(-1)}\delta_{KN}.$$

when $K = N$, we have $E_{(-1)} = 0$. And when $K \neq N$, we obtain $a_K^{(1)} = 0$. Note that this equation does not determine $a_N^{(1)} \equiv a_1(y)$. So we have

$$\psi_1 = a_1(y)\chi_N, \quad E_{(-1)} = 0. \tag{61}$$

From equation (51), (57), (61) we have

$$(-\frac{1}{2}\partial_\xi^2 - E_N)\psi_2 + \chi_N(\hat{h} - E_{(0)})a_0 = 0.$$

We expand ψ_2 as

$$\psi_2 = \sum_M a_M^{(2)}(y)\chi_M(\xi).$$

and by putting this into the previous equation and by using (58), we obtain

$$a_K^{(2)}(E_K - E_N) = -(\hat{h} - E_{(0)})a_0\delta_{KN}.$$

when $K = N$, we have $(\hat{h} - E_{(0)})a_0 = 0$. And when $K \neq N$, we obtain $a_K^{(2)} = 0$. Just like (61) this equation does not determine $a_N^{(2)} \equiv a_2(y)$. So we have

$$\psi_2 = a_2(y)\chi_N(\xi). \quad a_0 = \varphi_n, \quad E_{(0)} = \lambda_n. \tag{62}$$

From equation (52), (57), (61) , (62), we have

$$(-\frac{1}{2}\partial_\xi^2 - E_N)\psi_3 = \chi_N(\lambda_n - \hat{h})a_1 + \chi_N(E_{(1)}\varphi_n - \xi\hat{f}\varphi_n).$$

We expand ψ_3 as

$$\psi_3 = \sum_M a_M^{(3)}(y)\chi_M(\xi).$$

By putting this into the previous equation and by using (58), we obtain

$$a_K^{(3)}(E_K - E_N) = -(\hat{h} - \lambda_n)a_1\delta_{KN} + \delta_{KN}E_{(1)}\varphi_n - < K|\xi|N > \hat{f}\varphi_n,$$

where

$$< K|\xi|N > \equiv \int \chi_K \ \xi \ \chi_N \ d\xi.$$

When $K \neq N$, we obtain

$$a_K^{(3)} = -\frac{< K|\xi|N >}{E_K - E_N}\hat{f}\varphi_n \quad (K \neq N).$$

When $K = N$, we have

$$-(\hat{h} - \lambda_n)a_1 + E_{(1)}\varphi_n = < N|\xi|N > \hat{f}\varphi_n = 0.$$

Last equality comes from $< N|\xi|N > = 0$. By using the expansion

$$a_1 = \sum_l^\infty b_l^{(1)}\varphi_l,$$

multiplying $\sqrt{g}\varphi_m^*$, and integrating by y, we obtain from (60)

$$0 = b_m^{(1)}(\lambda_n - \lambda_m) + E_{(1)}\delta_{nm}.$$

Then we have

$$b_m^{(1)} = 0 \ (m \neq n), \quad E_{(1)} = 0.$$

Note that $b_n^{(1)}$ still remains as undetermined constant, however, such term appears as

$$\psi_1 = b_n^{(1)}\varphi_n\chi_N.$$

This is the same function as ψ_0 in expansion. So we take $b_n^{(1)} = 0$ in the following. In total we obtain

$$\begin{aligned} \psi_3 &= a_3\chi_N - \sum_{K\neq N}\frac{< K|\xi|N >}{E_K - E_N}\chi_K \ \hat{f}\varphi_n, \\ &\psi_1 = 0, \quad E_{(1)} = 0. \end{aligned} \tag{63}$$

From equation (53),(57),(61),(62),(63), we have

$$-\frac{1}{2}\partial_\xi^2\psi_4 + \hat{h}a_2\chi_N + \xi^2\hat{g}\varphi_n\chi_N = E_{(2)}\varphi_n\chi_N + \lambda_n a_2\chi_N + E_N\psi_4.$$

We expand ψ_4 as

$$\psi_4 = \sum_M a_M^{(4)}(y)\chi_M(\xi).$$

By putting this expansion into the previous equation and by using (58), we obtain

$$a_K^{(4)}(E_K - E_N) + \delta_{KN}(\hat{h} - \lambda_n)a_2 + < K|\xi^2|N > \hat{g}\varphi_n - \delta_{KN}E_{(2)}\varphi_n = 0,$$

For $K \neq N$, we obtain

$$a_K^{(4)} = -\frac{< K|\xi^2|N >}{E_K - E_N}\hat{g}\varphi_n \quad (K \neq N).$$

For $K = N$, we obtain just like before

$$a_2 = -\sum_{m\neq n}\frac{< N|\xi^2|N >}{\lambda_m - \lambda_n}(m|\hat{g}|n)\varphi_m,$$

$$E_{(2)} = < N|\xi^2|N > (n|\hat{g}|n),$$

where

$$(m|\hat{g}|n) \equiv \int \sqrt{g} \ \ \phi_m^*\hat{g}\phi_n d^2y.$$

Then we obtain

$$\begin{aligned} \psi_4 &= a_4\chi_N - \sum_{K\neq N}\frac{< K|\xi^2|N >}{E_K - E_N}\chi_K \ \hat{g}\varphi_n, \\ \psi_2 &= -\sum_{m\neq n}\frac{< N|\xi^2|N >}{\lambda_m - \lambda_n}(m|\hat{g}|n)\varphi_m \ \chi_N, \\ E_{(2)} &= < N|\xi^2|N > (n|\hat{g}|n). \end{aligned} \tag{64}$$

In total we obtain the wave function and energy eigenvalue up to $\mathcal{O}(\epsilon^2)$.

$$\begin{aligned} \tilde{\psi} &= \{\varphi_n - \epsilon^2\sum_{m\neq n}\frac{< N|\xi^2|N >}{\lambda_m - \lambda_n}(m|\hat{g}|n)\varphi_m\} \\ &\qquad \times \ \chi_N \ \exp(-iEt/\hbar), \end{aligned} \tag{65}$$

$$E = \epsilon^{-2}E_N + \lambda_n + \epsilon^2 < N|\xi^2|N > (n|\hat{g}|n), \tag{66}$$

where we brought out the explicit time dependence. Note that we have no N number excitation, and separation of variable method holds up to $\mathcal{O}(\epsilon^2)$.

The effective Hamiltonian that leads to (65), (66) is

$$\hat{H}_{eff}^N = -\frac{1}{2\epsilon^2}\partial_\xi^2 + \hat{h} + \epsilon^2 < N|\xi^2|N > \hat{g}.$$

with a solution in the form of $\tilde{\psi} = \varphi(t) \ \chi_N(t)$. Then the Schrödinger equation for φ field is

$$i\hbar\frac{\partial\varphi}{\partial t} = [\ \hat{h} + \epsilon^2 < N|\xi^2|N > \hat{g} \ ] \ \varphi. \tag{67}$$

In the original variable we have

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\{\Delta^{(2)} + V_0 + \tilde{\epsilon}^2(V_2 + \hat{A}_2)\}\varphi + V\varphi, \quad (68)$$

where

$$\tilde{\epsilon}^2 \equiv \epsilon^2 \langle N|\, \xi^2\, |N\rangle = \frac{\epsilon^2}{12}(1 - \frac{6}{\pi^2 N^2}). \quad (69)$$

From this Schrödinger equation, we obtain

$$\begin{aligned} -\ \frac{\partial}{\partial t}|\varphi|^2 &= \frac{\hbar}{2mi}\{(\varphi^*\Delta^{(2)}\varphi - \varphi\Delta^{(2)}\varphi^*) \\ &\quad + \tilde{\epsilon}^2(\varphi^*\hat{A}_2\varphi - \varphi\hat{A}_2\varphi^*)\} \\ &= \frac{\hbar}{2mi}\nabla_i\{(g^{ij} + 3\tilde{\epsilon}^2\kappa^{ik}\kappa_k^j)(\varphi^*\partial_j\varphi - \varphi\partial_j\varphi^*)\} \\ &= \nabla_i(J^i + J^i_G), \end{aligned} \quad (70)$$

where

$$\begin{aligned} J^i &\equiv \frac{\hbar}{2mi}g^{ij}(\varphi^*\partial_j\varphi - \varphi\partial_j\varphi^*), && (71) \\ J^i_G &\equiv \frac{3\hbar\tilde{\epsilon}^2}{2mi}\ \kappa^{ik}\kappa_k^j\ (\varphi^*\partial_j\varphi - \varphi\partial_j\varphi^*). && (72) \end{aligned}$$

Note that the back reaction from φ field to the χ field occurs in $\mathcal{O}(\epsilon^3)$ (See (63) and (64)).

5. Toy model

To consider the physical meaning of geometric flow, let us show one simple example: bending ribbon with thickness ϵ as seen in figure 2.

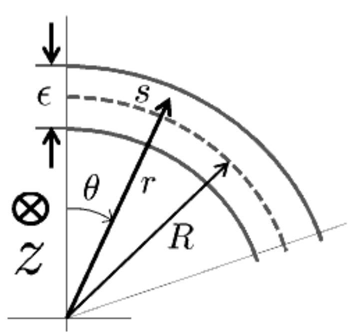

FIG. 2: Bending ribbon (side view) with thickness ϵ.

The physical space is inner ribbon with $0 < z < L$, and $-\epsilon/2 < r - R < +\epsilon/2$. Our starting equation is an usual three dimensional Schrödinger equation (20). Then we obtain the conservation law,

$$\nabla_\mu \tilde{J}^\mu = -\frac{\partial\tilde{\rho}}{\partial t}, \quad (73)$$

$$\tilde{J}^\mu = \frac{\hbar}{2mi}G^{\mu\nu}(\psi^*\nabla_\nu\psi - \psi\nabla_\nu\psi^*), \quad (74)$$

where $\tilde{J}$ and $\tilde{\rho}$ are three dimensional flow and density respectively. We utilize the cylindrical coordinates (r, θ, z) and

$$\begin{aligned} ds^2_{(3)} &= dr^2 + r^2 d\theta^2 + dz^2, && (75) \\ G_{\mu\nu} &= \mathrm{diag}(1, r^2, 1), \sqrt{G} = r. && (76) \end{aligned}$$

$$\begin{aligned} \tilde{J}^r &= \frac{\hbar}{2mi}(\psi^*\partial_r\psi - \psi\partial_r\psi^*), && (77) \\ \tilde{J}^\theta &= \frac{\hbar}{2mr^2 i}(\psi^*\partial_\theta\psi - \psi\partial_\theta\psi^*), && (78) \\ \tilde{J}^z &= \frac{\hbar}{2mi}(\psi^*\partial_z\psi - \psi\partial_z\psi^*). && (79) \end{aligned}$$

Then the conservation law gives

$$\partial_\theta\tilde{J}^\theta + \partial_r\tilde{J}^r + \partial_z\tilde{J}^z + \frac{1}{r}\tilde{J}^r = -\partial_t\tilde{\rho}. \quad (80)$$

The volume element is given by

$$dV = r dr d\theta dz = (r/R) dr ds dz, \quad (81)$$

where $ds = Rd\theta$. We then integrate both hand sides of (80) by $(r/R)dr$ in a region $R - \epsilon/2 \sim R + \epsilon/2$ and then we obtain two dimensional conservation law.

$$\partial_s J^s_{tot} + \partial_z J^z_{tot} = -\partial_t\rho, \qquad \int \rho\ ds dz = 1, \quad (82)$$

where

$$J^s_{tot} \equiv \int_{R-\epsilon/2}^{R+\epsilon/2} r\tilde{J}^\theta\ dr, \quad (83)$$

$$J^z_{tot} \equiv \int_{R-\epsilon/2}^{R+\epsilon/2} r\tilde{J}^z/R\ dr, \quad (84)$$

$$\rho \equiv \int_{R-\epsilon/2}^{R+\epsilon/2} r\tilde{\rho}/R\ dr, \quad (85)$$

where the boundary condition $\tilde{J}^r = 0$ at $r = R \pm \epsilon/2$ and $ds = Rd\theta$ are utilized. We have

$$J^s_{tot} = \frac{\hbar}{2mi}\int_{R-\epsilon/2}^{R+\epsilon/2} \frac{dr}{r}(\psi^*\partial_\theta\psi - \psi\partial_\theta\psi^*). \quad (86)$$

The separation of variable (24), (26) gives

$$\psi(r, \theta, z) = (g/G)^{1/4}\varphi(\theta, z)\chi(r) = \sqrt{\frac{R}{r}}\varphi(\theta, z)\chi(r), \quad (87)$$

where

$$\int dr\ |\chi|^2 = 1, \quad \int Rd\theta dz\ |\varphi|^2 = \int ds dz\ |\varphi|^2 = 1,$$

$$ds^2_{(2)} = R^2 d\theta^2 + dz^2, \quad g_{ij} = \mathrm{diag}(R^2, 1), \quad \sqrt{g} = R.$$

Then we obtain

$$J^s_{tot} = \frac{\hbar R^2}{2mi}\int_{R-\epsilon/2}^{R+\epsilon/2}\frac{dr}{r^2}\ |\chi(r)|^2(\varphi^*\partial_s\varphi - \varphi\partial_s\varphi^*). \quad (88)$$

Equation (54) gives the explicit form of χ. By using

$$\xi = q^0/\epsilon = (r-R)/\epsilon = Rx/\epsilon,$$

we obtain

$$\begin{aligned} J^s_{tot} &= \frac{\hbar R}{mi\epsilon}\int_{-\epsilon/2R}^{+\epsilon/2R}\frac{dx}{(1+x)^2}\cos^2((N\pi R/\epsilon)x), \\ &\qquad \times(\varphi^*\partial_s\varphi - \varphi\partial_s\varphi^*), \end{aligned} \quad (89)$$

for odd N. For even N we just change cos to sin.

We expand the integrand as

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 + \cdots. \quad (90)$$

Then we obtain

$$J^s_{tot} = \frac{\hbar}{2mi}(1 + 3\tilde{\epsilon}^2/R^2 + \cdots)(\varphi^*\partial_s\varphi - \varphi\partial_s\varphi^*), \quad (91)$$

for both of even and odd N. If we use coordinate $q^i = (s, z)$, we have $g_{ij} = \delta_{ij}$, $\kappa_{ss} = 1/R$, $\kappa_{sz} = \kappa_{zz} = 0$. Then we find this is completely equal to $J^s + J^s_G$ appeared in (71) and (72).

On the other hand,

$$\begin{aligned} J^z_{tot} &= \frac{\hbar}{2mi}(\int_{R-\epsilon/2}^{R+\epsilon/2} dr\ |\chi(r)|^2)(\varphi^*\partial_z\varphi - \varphi\partial_z\varphi^*) \\ &= \frac{\hbar}{2mi}(\varphi^*\partial_z\varphi - \varphi\partial_z\varphi^*). \end{aligned} \quad (92)$$

This is equal to J^z in (71) , and we have no geometric flow in this straight direction. Geometric flow is contained as the part of the integrand in equation (88): the second order of expansion (90). We can transcribe (88) into the form

$$J^s_{tot} = \frac{\hbar}{2mi} < (\frac{R}{r})^2 > (\varphi^*\partial_s\varphi - \varphi\partial_s\varphi^*), \quad (93)$$

$$< f >_{qm} \equiv \int_{R-\epsilon/2}^{R+\epsilon/2} f(r)\ |\chi(r)|^2 dr. \quad (94)$$

5. Conclusion

We have discussed the conservation law in an effective two-dimensional system between two curved surfaces Σ' and $\tilde{\Sigma}$ separated by a small distance ϵ. We found that the anomalous flow depends on the curvature of the surface Σ. In the classical diffusion process, we have

$$J^i_G = -\frac{\epsilon^2 D}{12}\ [\ (3\kappa^{im}\kappa^j_m - 2\kappa\kappa^{ij})\frac{\partial\phi^{(2)}}{\partial q^j} - \frac{1}{2}g^{ij}\frac{\partial R}{\partial q^j}\phi^{(2)}] \quad (95)$$

as shown in [2].

In the quantum process we instead obtain

$$J^i_G = \frac{\hbar\epsilon^2}{8mi}(1 - \frac{6}{\pi^2 N^2})\ \kappa^{ik}\kappa^j_k\ (\varphi^*\partial_j\varphi - \varphi\partial_j\varphi^*). \quad (96)$$

The classical anomalous flow and quantum mechanical anomalous flow are somewhat similar. Both start from $\mathcal{O}(\epsilon^2\kappa^2)$ and are proportional to the gradient of the field except the last term in the classical flow. The recent nano-technology made it possible to fabricate complicate devices. Then this kind of anomalous flow might play an important role in such a "geometrical" device.

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